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Cartan–Eilenberg cohomology and triples

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Abstract

In their classic book, Cartan and Eilenberg described a more-or-less general scheme for defining homology and cohomology theories for a number of different kinds of algebraic structure, using a general theory of augmented algebras. Later, in his doctoral dissertation, Beck showed how to use the theory of triples to derive a very different and completely general scheme for doing the same thing. Originally, it was unclear how the two theories were related, but many of these questions were eventually answered in a paper by Barr and Beck. The present paper answers the remaining such questions, most notably in the case of Lie algebras by finding a general result that takes care of all the cases at once. It also shows that it is possible to extend the Cartan–Eilenberg theory of Lie algebras from algebras that are free over the ground ring to ones that are only projective.

1. Introduction

The genesis of this paper is in [1] in which it is shown that, with a shift in dimension, the cohomology theories for groups and associative algebras of [4] were the same as those that were introduced by Beck in his doctoral dissertation [3] and could thus be viewed as the derived functors of the derivations functor on the categories in question and computed by "cotriple resolutions" in those categories. This was the first use of acyclic models as a tool in algebraic cohomology theories. In that paper, we never examined the case of Lie algebras and, oddly enough, this gap has not been filled in the intervening time.

In this paper, we show that the results of [1] hold in some generality, sufficient to include the case of Lie algebras and many others in the Cartan–Eilenberg (CE) context. As a minor bonus, we show that the hypothesis of Chapter XIII of [4] on Lie algebras,

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that the algebra be free over the ground ring, can be relaxed to the assumption that the algebra is projective.

1.1. The dimension shift. There is one unvarying feature of these comparisons. The CE theory begins one dimension lower than that of Beck. Let us write H_n^{B} and H_B^n for the homology and cohomology theories of Beck and write H_n^{CE} and H_{CE}^n for those of [4]. Then the comparison theorems are between H_{B}^n and H_{CE}^{n+1} and there is nothing in the Beck theory corresponding to H_{CE}^0.

1.2. Notation for identity maps. Sometimes we denote the identity map of some object by id and sometimes by 1, depending on which is more convenient. The latter is used mostly for matrix entries and in computations.

2. Beck's theory

2.1. Modules. In his 1967 dissertation, Beck answered the question, "What is a module?" His answer was appropriate for the kind of module that was a coefficient module for cohomology. For groups, commutative algebras and Lie algebras, these are left modules; for associative algebras, the appropriate notion is that of two-sided module. Beck's definition captures exactly those notions in the various categories. This means that it encompasses all known notions of module, except, oddly enough, that of left module over an associative ring.

The basic idea is to identify a module with the split extension that has module as kernel. For example, if K is a commutative ring, A is an associative K-algebra and M is a two sided A-module (which includes the assumption that left and right multiplication on M by an element of K are the same), then the split extension is the K-algebra B that is, as a K-module, just $A \times M$ and whose multiplication is given by $(a,m)(a',m') = (aa', am' + ma')$. It turns out that the B that arise in this way can be characterized as the abelian group objects in the slice category $\mathscr{A}/A$, where $\mathscr{A}$ is the category of associative K-algebras.

Accordingly, we will define for an object A of a category $\mathscr{A}$ the category $\mathbf{Mod}(A)$ as the category of abelian group objects of the slice $\mathscr{A}/A$.

Here we show that works for associative K-algebras. Let A, M and B be as above. Then an abelian group object of a category is determined by certain arrows, namely, a zero map $1 \to B$, an inverse map $B \to B$ and a group multiplication $B \times B \to B$. In the slice category, the terminal object is A and that product is the fibered product $B \times_A B$. The zero map takes the element a to $(a, 0)$, the inverse map is given by $(a,m) \mapsto (a,-m)$ and the multiplication takes the pair $((a,m),(a,m'))$ in the fiber over a to the element $(a, m+m')$. This makes B into an abelian group in $\mathscr{A}/A$. The inverse functor takes $B \to A$ to the kernel of that arrow, which turns out to be an A-module in the usual sense. The details are found in [3] but we give a sketch of the argument in an appendix to this paper.

2.2. Regular epimorphisms and regular categories. This section records some facts about regular epimorphisms and regular categories. Details and proofs can be found in [2, Chapter 1].

An arrow in a category is called a *regular epimorphism* (often abbreviated "regular epi" if it is a coequalizer of two arrows into its domain. If the arrow has a kernel pair (the pullback of the arrow with itself), then it is a regular epi if and only if it is the coequalizer of that kernel pair. A category is a *regular category* if in any pullback diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "f"] & D \end{tikzcd}$$

whenever f is a regular epi, so is g. Among the many properties of regular epis is that they are *strict*, which means they factor through no proper subobject of their codomain.

Proposition 2.3. *Let $\mathscr{A}$ is a regular category. Then the forgetful functor I_A :* **Mod** *$(A) \to \mathscr{A}/A$ preserves regular epis.*

Proof. What we have to show is that if $f : M' \to M$ is a regular epimorphism in the category **Mod**(A), then it is also a regular epi in $\mathscr{A}/A$. Actually, we will show that if f is a strict epi in **Mod**(A), then it is regular in $\mathscr{A}$ and hence in $\mathscr{A}/A$.

An object of **Mod**(A) is an object $B \to A$ equipped with certain arrows of which the most important is the arrow $m : B \times_A B \to B$ that defines the addition. There are also some equations to be satisfied. The argument we give actually works in the generality of the models of a finitary equational theory. So suppose $f : M' \twoheadrightarrow M$ is a strict epimorphism in **Mod**(A). If the map $I_A f$ is not a strict epi, it can be factored as $B' = I_A M' \twoheadrightarrow B'' \rightarrowtail B = I_A M$ in $\mathscr{A}/A$. Since $\mathscr{A}$, and hence $\mathscr{A}/A$, is regular, the arrow $B' \times_A B' \to B'' \times_A B''$ is also regular epic and we have the commutative diagram

$$\begin{tikzcd} B' \times_A B' \arrow[r, two heads] \arrow[d, "m'"'] & B'' \times_A B'' \arrow[r, tail] & B \times_A B \arrow[d, "m"] \\ B' \arrow[r, two heads] & B'' \arrow[r, tail] & B \end{tikzcd}$$

The "diagonal fill-in" (here vertical) provides the required arrow $m'' : B'' \times_A B'' \to B''$ at the same time showing that both of the arrows $B' \to B'' \to B$ preserve the new operation. A similar argument works for any other finitary operation. As for the equations that have to be satisfied, this follows from the usual argument that shows that subcategories defined by equations are closed under the formation of subobjects. For example, we show that m'' is associative. This requires showing that two arrows $B'' \times_A B'' \times_A B'' \to B''$ are the same. But we have the diagram

$$\begin{tikzcd} B'' \times_A B'' \times_A B'' \arrow[r, tail] \arrow[d, shift left] \arrow[d, shift right] & B \times_A B \times_A B \arrow[d] \\ B'' \arrow[r, tail] & B \end{tikzcd}$$

that commutes with either of the two left hand arrows. With the bottom arrow monic, this means those two arrows are equal. □

In all cases, the category of modules is a category of modules over some commutative ring we call, following [4], A^e (for enveloping algebra of A). This follows immediately from the Morita theorems, which define A^e only up to Morita equivalence. The (co)homology theories are actually invariant to Morita equivalence, but there does in fact always seem to be a natural choice for A^e.

2.4. Derivations. At the same time, Beck answered the question of what is a derivation. If an A-module is an abelian group object in $\mathscr{A}/A$, then for any A-module M and any $B \to A$, the hom set $\mathscr{A}/A(B, I_A M)$ is an abelian group. In all the traditional cases – groups, Lie, associative and commutative algebras – the abelian group is the group of derivations of B to M, where the action of B on M is induced from that of A by the arrow $B \to A$.

In the cases of interest to us, the inclusion I_A has a left adjoint. In fact, it is not hard to show that this adjoint necessarily exists when $\mathscr{A}$ is locally presentable in the sense of [5]. Suppose we temporarily call this adjoint J^A. Then for any $B \to A$ and any A-module M, we have

$$\mathbf{Mod}(A)(J^A B, M) \cong \mathscr{A}/A(B, I_A M) \cong \mathrm{Der}(B, M)$$

so that J^A represents the functor Der. For this reason, we call $J^A(B)$ the A-module of differentials on B and denote it henceforth by $\mathrm{Diff}^A(B)$.

2.5. Cotriple homology and cohomology. Beck went on from the definition of module, derivations and differentials to define homology and cohomology theories that we describe briefly here. In all these cases, there is a cotriple $\mathbf{G} = (G, \varepsilon, \delta)$ on $\mathscr{A}$ that comes from the composite of an underlying and a free functor. In the case of groups, the underlying functor is to the category of sets and in all other cases, we are dealing with a category of K-algebras for a commutative ring K and the underlying functor is to K-modules. Beck created the simplicial resolution

$$\cdots \begin{smallmatrix}\to\\ \vdots \\ \to\end{smallmatrix} G^{m+1}A \begin{smallmatrix}\to\\ \vdots \\ \to\end{smallmatrix} \cdots \begin{smallmatrix}\to\\ \to \\ \to\end{smallmatrix} G^2 A \rightrightarrows GA$$

in the category $\mathscr{A}/A$. He then defined homology as the homology of the simplicial object got by applying the functor $\mathrm{Diff}^A(-) \otimes M$ to the resolution above. The tensor product is that of the category of A-modules (or A^{e}-modules). The cohomology comes in a similar way by applying the contravariant functor $\mathrm{Der}(-,M)$, for an A-module M to get a cosimplicial object

$$0 \to \mathrm{Der}(GA,M) \rightrightarrows \mathrm{Der}(G^2A,M) \overrightarrow{\underrightarrow{\to}} \cdots \overrightarrow{\underrightarrow{\vdots}} \mathrm{Der}(G^{n+1}A,M) \overrightarrow{\underrightarrow{\vdots}} \cdots$$

and taking the cohomology.

3. The Cartan–Eilenberg setting

In [4], Cartan and Eilenberg give a more or less uniform definition of cohomology theories in various algebraic categories that can be described as follows. In each of the various categories, they associate to each object the enveloping associative algebra A^{e}, as previously described together with a left A^{e}-module we call $Z(A)$. Then for any A-module M, the homology and cohomology are defined as $\mathrm{Tor}_{\bullet}^{A^{\mathrm{e}}}(M,Z(A))$ and $\mathrm{Ext}_{A^{\mathrm{e}}}^{\bullet}(Z(A),M)$, respectively. One point to note here is that the Tor is defined on right A^{e}-modules and the cohomology on left A^{e}-modules. However, A^{e} is always isomorphic to its opposite ring and the three notions of A-module (in the sense of Beck), left A^{e}-module and right A^{e}-module coincide. For example, in the case of associative rings, an A-module in the sense of Beck is a two-sided A in the usual sense and if M is such a module, it is a left $A^{\mathrm{e}} = A \otimes A^{\mathrm{op}}$-module according to $(a \otimes b)m = amb$ and a right A^{e}-module by defining $m(a \otimes b) = bma$.

Actually, this description of the Cartan–Eilenberg definitions is somewhat misleading. They actually construct a "standard projective resolution" $C_{\bullet}(A)$ of $Z(A)$ that can be used to compute the Tor and Ext above. This standard resolution allows us to compare the Cartan–Eilenberg theory with the cotriple theory.

There are two apparently *ad hoc* elements in this definition. The first is the definition of module (and therefore of the enveloping algebra) and the second is the definition of $Z(A)$. Cartan and Eilenberg simply give them, with no attempt to find a systematic basis for describing them. Beck solved the first problem and, indirectly, the second. We have already described how Beck solved the problem of how to systematically describe a category of modules.

As for the second, the key is actually in the standard complex. Let us denote it by

$$\cdots \to C_n(A) \to \cdots \to C_1(A) \to C_0(A) \to Z(A) \to 0.$$

The module $Z(A)$ is rather arbitrary, but in every case, the kernel of $C_0(A) \to Z(A)$ is the module $\mathrm{Diff}^A(A)$. This means that there is an exact sequence

$$\cdots \to C_n(A) \to \cdots \to C_2(A) \to C_1(A) \to \mathrm{Diff}^A(A) \to 0$$

which is a projective resolution of $\mathrm{Diff}^A(A)$. Moreover, $\mathrm{Diff}^A(A)$ can be described in an intrinsic way, as we have already pointed out.

Define the *shifted Cartan–Eilenberg* (co)homology as $H_\bullet^{\mathrm{CES}}(A,M) = \mathrm{Tor}_\bullet(M, \mathrm{Diff}^A(A))$ and $H^\bullet_{\mathrm{CES}}(A,M) = \mathrm{Ext}^\bullet(\mathrm{Diff}^A(A), M)$. The connection between the shifted and original theories is stated in the following proposition, whose proof is trivial.

Proposition 3.1. *For any object A of one of the Cartan–Eilenberg categories and any A-module M, $H_n^{\mathrm{CES}}(A,M) \cong H_{n+1}^{\mathrm{CE}}(A,M)$, for $n > 0$, $H_0^{\mathrm{CES}}(A,M) \cong M \otimes_{A^e} \mathrm{Diff}^A(A)$ and $H_1^{\mathrm{CE}}(A,M)$ is a subgroup of $M \otimes_{A^e} \mathrm{Diff}^A(A)$. Similarly, $H^n_{\mathrm{CES}}(A,M) \cong H^{n+1}_{\mathrm{CE}}(A,M)$, for $n > 0$, $H^0_{\mathrm{CES}}(A,M) \cong \mathrm{Der}(A,M)$ and $H^1_{\mathrm{CE}}(A,M)$ is a quotient group of $\mathrm{Der}(A,M)$.*

What we will be showing is that, in the appropriate setting, the cotriple (co)homology groups are equivalent to the shifted Cartan–Eilenberg groups.

3.2. The standard setting. In order to understand these things in some detail, we describe what we call a *standard Cartan–Eilenberg* or *CE setting*.

We begin with a regular category $\mathscr{A}$. For each object A of $\mathscr{A}$, we denote by $\mathbf{Mod}(A)$ the category $\mathbf{Ab}(\mathscr{A}/A)$ of abelian group objects of $\mathscr{A}/A$. We assume that the inclusion $I_A : \mathbf{Mod}(A) \to \mathscr{A}/A$ has a left adjoint we denote Diff^A. When $f : B \to A$ is an arrow of $\mathscr{A}$, the direct image (or composite with f) determines a functor $f_! : \mathscr{A}/B \to \mathscr{A}/A$ that has a right adjoint $f^* = B \times_A -$ of pulling back along $B \to A$. The right adjoint (but not the direct image) induces a functor, we will also denote by $f^* : \mathbf{Mod}(A) \to \mathbf{Mod}(B)$ that we will assume has a left adjoint we will denote $f_\#$. The diagram is

$$\begin{tikzcd}
\mathscr{A}/A \arrow[r, shift left, "\mathrm{Diff}^A"] \arrow[d, shift left, "f^*"] & \mathrm{Mod}\,(A) \arrow[l, shift left, "I_A"] \arrow[d, shift left, "f^*"] \\
\mathscr{A}/B \arrow[r, shift left, "\mathrm{Diff}^B"] \arrow[u, shift left, "f_!"] & \mathrm{Mod}\,(B) \arrow[l, shift left, "I_B"] \arrow[u, shift left, "f_\#"]
\end{tikzcd}$$

The upper and left arrows are left adjoint, respectively, to the lower and right arrows and the diagram of the right adjoints commutes, and therefore, does the diagram of left adjoints. The left adjoint $f_\#$ turns out to be the functor $A^e \otimes_{B^e} (-)$. That is, A^e becomes a right B^e-module via f (actually, just the right hand version of f^*) and then that tensor product is an A^e-module.

We assume, given a base category $\mathscr{X}$ and an underlying functor $U : \mathscr{A} \to \mathscr{X}$ that preserves regular epis and has a left adjoint F. Let $\mathbf{G} = (G, \varepsilon, \delta)$ denote the resultant cotriple on $\mathscr{A}$.

We suppose, there is given for each object A of $\mathscr{A}$, a chain complex functor $C_\bullet^A : \mathscr{A}/A \to \mathbf{ChCompMod}(A)$, the category of chain complexes in $\mathbf{Mod}(A)$. That is, it assigns to each $B \to A$ a chain complex

$$\cdots \to C_n^A(B) \to \cdots \to C_0^A(B) \to 0$$

of A-modules. We further suppose that for $f : B \to A$ the diagram

$$\begin{tikzcd}
\mathscr{A}/A \arrow[r, "C_\bullet^A"] & \mathrm{ChCompMod}\,(A) \\
\mathscr{A}/B \arrow[u, "f_!"] \arrow[r, "C_\bullet^B"] & \mathrm{ChCompMod}\,(B) \arrow[u, "\mathrm{ChComp}\, f_\#"']
\end{tikzcd}$$

commutes.

Note that all these categories have initial objects. If we take B to be the initial object, then we get a standard complex for that case and the complex in all the other cases is got by applying $i_\#$, where i is the initial morphism. In light of a previous remark, this is just tensoring with A^e.

4. The main theorem

For the purposes of this theorem, define an object A of $\mathscr{A}$ to be *U-projective* if UA is projective in $\mathscr{X}$ with respect to the class of regular epis.

Theorem 4.1. *Suppose that, in the context of a CE setting, when A is U-projective,*

(i) *GA is U-projective;*

(ii) *$C_\bullet^A(A)$ is a projective resolution of* $\mathrm{Diff}^A(A)$;

(iii) *For each $n \geq 0$, there is a functor $\tilde{C}_n^A : \mathscr{X}/UA \to \mathbf{Mod}(A)$ such that the diagram*

$$\begin{tikzcd}
\mathscr{A}/A \arrow[r, "C_n^A"] \arrow[d, "U/A"'] & \mathrm{Mod}\,(A) \\
\mathscr{X}/UA \arrow[ur, "\tilde{C}_n^A"'] &
\end{tikzcd}$$

commutes. Then the complexes $C_\bullet^A(A)$ and $\mathrm{Diff}^A(G^{\bullet+1}A)$ *are chain equivalent.*

The last condition means that the modules in the projective resolution depend only on the object underlying A. Only the face operators depend on the actual structure. In the proof below, we fix A and write C_n and Diff for C_n^A and Diff^A, respectively.

Proof. We prove this by applying a general theorem on double complexes that we defer to the end of the paper (Corollary 6.7). To apply this theorem, we must show that in the double complex

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & C_nG^{m+1}A \arrow[r] \arrow[d] & \cdots \arrow[r] & C_0G^{m+1}A \arrow[r] \arrow[d] & \mathrm{Diff}G^{m+1}A \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & C_nGA \arrow[r] \arrow[d] & \cdots \arrow[r] & C_0GA \arrow[r] \arrow[d] & \mathrm{Diff}GA \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & C_nA \arrow[r] \arrow[d] & \cdots \arrow[r] & C_0A \arrow[r] \arrow[d] & \mathrm{Diff}A \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 & 0 &
\end{tikzcd}$$

all rows except the bottom and all columns except the right hand one are contractible.
The column

$$\cdots \to C_nG^{m+1}A \to \cdots \to C_nGA \to C_nA \to 0$$

is equivalent to

$$\cdots \to \tilde{C}_nUG^{m+1}A \to \cdots \to \tilde{C}_nUGA \to \tilde{C}_nUA \to 0.$$

At this point we require,

Lemma 4.2. *Let the functor $U : \mathscr{A} \to \mathscr{X}$ have left adjoint F and let $\mathbf{G}$ be the resultant cotriple on $\mathscr{A}$. Then for any object A of $\mathscr{A}$, the simplicial object*

$$\cdots \substack{\to\\ \vdots\\ \to}\ UG^{m+1}A\ \substack{\to\\ \vdots\\ \to} \cdots \substack{\to\\ \to\\ \to}\ UG^2A \rightrightarrows UGA \to UA$$

is contractible.

Proof. We let $s = \eta UG^nA : UG^mA \to UG^{m+1}A$. Then

$$Ud^0 \circ s = U\varepsilon G^mA \circ \eta UG^mA = \mathrm{id}$$

while, for $0 < i \leq m$,

$$Ud^i \circ \eta UG^mA = UG^i\varepsilon G^{m-i}A \circ \eta UG^mA$$

and the last term equals, by naturality of η,

$$\eta UG^{m-1}A \circ UG^{i-1}\varepsilon G^{m-i}A = \eta UG^{m-1}A \circ Ud^{i-1}$$

This shows that s is a contracting homotopy in the simplicial object. □

If we apply the additive functor $\tilde{C}_n$ to this contractible complex, we still get a contractible complex, which shows that the columns of the double complex, except for the rightmost, are contractible. For the rows, we require the following.

Lemma 4.3. *Let P be a regular projective object of $\mathscr{X}$. Then, for any $P \to UA$,* $\mathrm{Diff}(FP)$ *is projective.*

Proof. When P is projective in $\mathscr{X}$, any $P \to X$ is a projective object of $\mathscr{X}/X$. It is immediate that when $L : \mathscr{X} \to \mathscr{Y}$ is left adjoint to $R : \mathscr{Y} \to \mathscr{X}$, then L takes a projective in $\mathscr{X}$ to a projective in $\mathscr{Y}$ provided R preserves the epimorphic class that defines the projectives. In this case, the right adjoint is the composite UI_A and the class is that of regular epimorphisms. We have assumed that U, and hence U/A, preserves regular epis and Proposition 2.3 says that I_A does.

With this lemma we see that the rows of the double complex, save for the bottom row, are projective resolutions of projectives modules and are, therefore, also contractible. This establishes the theorem. □

5. Applications

5.1. Groups. Let **Gp** be the category of groups and π be a group. The underlying functor $U : \mathbf{Gp} \to \mathbf{Set}$ evidently satisfies our conditions and the fact that epimorphisms in **Set** split implies that every group is U-projective. If we fix a group π, the functor $\tilde{C}_n^{\pi} : \mathbf{Set}/U\pi \to \mathbf{Mod}(\pi)$ takes the set $g : S \to U\pi$ to the free π-module generated by the $(n+1)$th cartesian power S^{n+1}. Now suppose that $g = Uf$ for a group homomorphism $f : \Pi \to \pi$. The value of the boundary operator ∂ on a generator $\langle x_0, x_1, \ldots, x_n \rangle \in \tilde{C}_n^{\pi}(Uf : U\Pi \to U\pi)$ is

$$f(x_0)\langle x_1, \ldots, x_n \rangle + \sum_{i=1}^{n-1} (-1)^i \langle x_0, \ldots, x_{i-1}x_i, \ldots, x_n \rangle + (-1)^n \langle x_0, \ldots, x_{n-1} \rangle$$

which depends on the group structure in Π. This defines the functor $C_{\bullet}^{\pi}$ on $\mathbf{Gp}/\pi$. The standard Cartan–Eilenberg resolution is the special case of this one in which f is the identity $\pi \to \pi$. We may denote $C_{\bullet}^{\pi}(\mathrm{id} : U\pi \to U\pi)$ as simply $C_{\bullet}(\pi)$ (U applied to the identity of π is the identity of $U\pi$). It is shown in [4] that $C_{\bullet}(\pi)$ is a projective resolution of $\mathrm{Diff}^{\pi}(\pi)$. More precisely, it is shown that the complex extended by one term is a projective resolution of $Z(\pi)$ which in this case is the group of integers with trivial action by π. Thus the conditions of (4.1) are satisfied and we conclude that the group cohomology is the cotriple cohomology.

Although $\tilde{C}_n^{\pi}(S \to U\pi)$ could, in principle, depend on the arrow $S \to U\pi$, in practice, in this example and the others it does not. The boundary operator does, however.

5.2. Associative algebras. The situation with associative algebras is quite similar. We begin with a commutative (unitary) ring K. The category $\mathscr{X}$ is the category of

K-modules and $\mathscr{A}$ is the category of K-algebras. If A is a K-algebra, the category **Mod**(A) is the category of two sided A-modules. The enveloping algebra of A is $A^{\mathrm{e}} = A \otimes_K A^{\mathrm{op}}$ and it is easy to see that two-sided A-modules are the same thing as left A^{e}-modules. The free algebra generated by a K-module M is the tensor algebra

$$F(M) = K \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \cdots \oplus M^{(n)} \oplus \cdots$$

and it is evident that $F(M)$ is K-projective when M is. Note that we use $M^{(n)}$ to denote the nth tensor power of M. If A is a K-algebra, the functor $\tilde{C}_n^A$ is defined by the formula

$$\tilde{C}_n^A(M \to UA) = A \otimes M^{(n+1)} \otimes A^{\mathrm{op}} \cong A^{\mathrm{e}} \otimes M^{(n+1)}$$

for $g : M \to UA$. The boundary formula is similar to the one for groups. If g has the form $Uf : UB \to UA$, then

$$\begin{aligned}\partial(a \otimes b_0 \otimes \cdots \otimes b_n \otimes a') &= af(b_0) \otimes b_1 \otimes \cdots \otimes b_n \otimes a' \\ &\quad + \sum_{i=1}^{n-1} (-1)^i a \otimes b_0 \otimes \cdots \otimes b_{i-1} b_i \otimes \cdots \otimes b_n \otimes a' \\ &\quad + (-1)^n a \otimes b_0 \otimes \cdots \otimes b_{n-1} \otimes f(b_n) a'\end{aligned}$$

differing only in the fact that we have operation on the right as well as on the left. The remaining details are essentially similar to those of the group case.

5.3. Lie algebras. This example differs from the preceding ones more than just in some details. For one thing, we would like to state a theorem for Lie algebras that are projective over the ground ring, not just free as done in [4]. For another, it is not clear that the free Lie algebra generated by a K-projective K-module is still K-projective. This fact is buried in an exercise in [4, Exercise 8 on p. 286], but is certainly not well-known, so we include the argument.

We begin by seeing what needs to be done to go from free modules to projectives. [4] makes use of this in two places. The first is in the Poincaré–Witt theorem, which states that the enveloping associative algebra generated by a K-free Lie algebra is K-free. The enveloping algebra in this comes from the adjoint, $\mathfrak{g} \mapsto \mathfrak{g}^{\mathrm{e}}$, to the "forgetful" functor from associative algebras to Lie algebras that replaces the multiplication in an associative algebra by the Lie bracket $[x, y] = xy - yx$. But if $\mathfrak{g}$ is K-projective, then we can find a K-module $\mathfrak{g}_0$ such that the K-module $\mathfrak{g} \oplus \mathfrak{g}_0$ is K-free. We can make $\mathfrak{g} \oplus \mathfrak{g}_0$ into a Lie algebra by making $\mathfrak{g}_0$ a central ideal (that is the product of any element of $\mathfrak{g}_0$ with any other element of the direct sum is 0). Then $\mathfrak{g}$ is, as a Lie algebra, a retract of $\mathfrak{g} \oplus \mathfrak{g}_0$. All functors preserve retracts so that $\mathfrak{g}^{\mathrm{e}}$ is a retract of $(\mathfrak{g} \oplus \mathfrak{g}_0)^{\mathrm{e}}$ and if the latter is K-free, then $\mathfrak{g}^{\mathrm{e}}$ is K-projective.

The second place that freeness is used in the theorem that if $\mathfrak{h}$ is a Lie subalgebra of the Lie algebra $\mathfrak{g}$, and if $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are K-free, then $\mathfrak{g}^{\mathrm{e}}$ is a free $\mathfrak{h}^{\mathrm{e}}$-module. We would like to prove this with "free" replaced everywhere by "projective".

Proposition 5.4. *Let* $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$ *be an exact sequence of K-projective K-Lie algebras. Then* $\mathfrak{g}^e$ *is projective as an* $\mathfrak{h}^e$*-module.*

Proof. The conclusion is valid when all three of $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are K-free [4, Proposition XIII.4.1]. For the general case, let $\tilde{\mathfrak{f}} = \mathfrak{g}/\mathfrak{h}$. Since $\mathfrak{h}$ is K-projective, there is a K-module $\mathfrak{h}_0$ such that $\mathfrak{h} \oplus \mathfrak{h}_0$ is K-free. If we give $\mathfrak{h}_0$ the structure of a central ideal, then $\mathfrak{h} \oplus \mathfrak{h}_0$ is a K-free K-Lie algebra. Similarly, choose $\mathfrak{f}_0$ so that $\tilde{\mathfrak{f}} \oplus \mathfrak{f}_0$ is K-free. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{h} \arrow[r, "f"] \arrow[d, "\begin{pmatrix}1\\0\end{pmatrix}"'] & \mathfrak{g} \arrow[r, "g"] \arrow[d, "\begin{pmatrix}1\\0\\0\end{pmatrix}"'] & \tilde{\mathfrak{f}} \arrow[r] \arrow[d, "\begin{pmatrix}1\\0\end{pmatrix}"] & 0 \\
0 \arrow[r] & \mathfrak{h} \oplus \mathfrak{h}_0 \arrow[r, "\begin{pmatrix}f&0\\0&1\\0&0\end{pmatrix}"] & \mathfrak{g} \oplus \mathfrak{h}_0 \oplus \mathfrak{f}_0 \arrow[r, "\begin{pmatrix}g&0&0\\0&0&1\end{pmatrix}"] & \tilde{\mathfrak{f}} \oplus \mathfrak{f}_0 \arrow[r] & 0
\end{tikzcd}$$

In the bottom sequence, the two ends are K-free from which it follows that the middle is as well. Apply the enveloping algebra functor to the left hand square to get the diagram, in which $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{f}_0$

$$\begin{tikzcd}
\mathfrak{h}^e \arrow[r] \arrow[d] & \mathfrak{g}^e \arrow[d] \\
(\mathfrak{h} \oplus \mathfrak{h}_0)^e \arrow[r] & (\mathfrak{g} \oplus \mathfrak{g}_0)^e
\end{tikzcd}$$

According to [4, Proposition XIII.2.1], for any two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, there is an isomorphism $(\mathfrak{g}_1 \oplus \mathfrak{g}_2)^e \cong \mathfrak{g}_1^e \otimes \mathfrak{g}_2^e$. This can be proved directly, as done previously, or by noting that both sides represent the functor that assigns to an associative algebra A the set of pairs of *pointwise commuting* homomorphisms in $\mathrm{Hom}(\mathfrak{g}_1^e, A) \times \mathrm{Hom}(\mathfrak{g}_2^e, A)$. Moreover, $\mathfrak{h}_0^e$ is K-projective since $\mathfrak{h}_0$ is. If $\mathfrak{h}_0^e$ is K-free, say $\mathfrak{h}_0^e \cong \sum K$, then $\mathfrak{h}^e \otimes \mathfrak{h}_0^e \cong \mathfrak{h}^e \otimes \sum K \cong \sum \mathfrak{h}^e$ is a free $\mathfrak{h}^e$-module. If $\mathfrak{h}_0^e$ is K-projective, then it is a retract of a free K-module and it follows that $\mathfrak{h}^e \otimes \mathfrak{h}_0^e$ is a retract of a free $\mathfrak{h}^e$-module. But $(\mathfrak{g} \oplus \mathfrak{g}_0)^e$ is, as an $(\mathfrak{h} \oplus \mathfrak{h}_0)^e$-module, *a fortiori* as an $\mathfrak{h}^e$-module, isomorphic to a direct sum of copies of $(\mathfrak{h} \oplus \mathfrak{h}_0)^e$ and hence is also $\mathfrak{h}^e$-projective. Finally, $\mathfrak{g}^e$ is a retract as a ring, therefore as a $\mathfrak{g}^e$-module and hence as an $\mathfrak{h}^e$-module, of $(\mathfrak{g} \oplus \mathfrak{g}_0)^e$ and is therefore also $\mathfrak{h}^e$-projective. □

With these two results, the entire Chapter XIII of [4] becomes valid with free replaced by projective.

Now we describe the standard theory from [4] (with the usual dimension shift). For a K-module M, let $\bigwedge^n(M)$ denote the nth exterior power of M. Then for a module homomorphism $g : M \to U\mathfrak{g}$, $\tilde{C}_n^{\mathfrak{g}}(M \to U\mathfrak{g}) = \mathfrak{g}^e \otimes \bigwedge^{n+1}(M)$. If $g = Uf$ for a Lie algebra homomorphism $f : \mathfrak{h} \to \mathfrak{g}$, the boundary is described on generators as follows, where, as usual, the $\hat{}$ denotes the omission of an argument.

$$\begin{aligned}\partial(x_0 \wedge x_1 \wedge \cdots \wedge x_n) &= \sum_{i=0}^{n} (-1)^i f(x_i) \otimes x_0 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \\ &\quad + \sum_{1 \le i < j \le n} (-1)^{i+j} [x_i, x_j] \wedge x_0 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n.\end{aligned}$$

In order to apply Theorem (4.1), we must show the following:

Proposition 5.5. *Let M be a projective K-module. Then the free Lie algebra FM is also K-projective.*

Proof (*Based on the hint to Exercise* 8 *of* [4, p. 286]). We consider first the case of a free K-module. There is a diagram of categories and adjoints

$$\begin{tikzcd}
\mathrm{Lie}\,(K) \arrow[rr, shift left] \arrow[d, shift left] & & \mathrm{Lie}\,(\mathbf{Z}) \arrow[ll, shift left, "K \otimes -"'] \arrow[d, shift left] \\
\mathrm{Mod}\,(K) \arrow[u, shift left] \arrow[rr, shift left] \arrow[dr, shift left] & & \mathrm{Mod}\,(\mathbf{Z}) \arrow[u, shift left] \arrow[ll, shift left, "K \otimes -"'] \arrow[dl, shift left] \\
& \mathrm{Set} \arrow[ul, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

It is clear from this diagram that if we show that the free $\mathbf{Z}$-Lie algebra generated by a free $\mathbf{Z}$-module (that is, abelian group) is a free abelian group, then by applying the functor $K \otimes_{\mathbf{Z}} -$, it follows that the free K-Lie algebra by a free K-module will be K-free.

So let M be a free abelian group and let $F(M)$ be the free Lie algebra generated by M. By the commutation of adjoints in the diagram

$$\begin{tikzcd}
\mathrm{Alg}\,(\mathbf{Z}) \arrow[rr, shift right] \arrow[dr, shift left] & & \mathrm{Lie}\,(\mathbf{Z}) \arrow[ll, shift right] \arrow[dl, shift left] \\
& \mathrm{Mod}\,(\mathbf{Z}) \arrow[ul, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

it follows that the enveloping associative algebra $F(M)^e$ is simply the tensor algebra $\mathbf{Z} \oplus M \oplus (M \otimes M) \oplus M^{(3)} \oplus \cdots$ which is $\mathbf{Z}$-free. The inner adjunction is a map $F(M) \to U(F(M)^e)$, where U is the forgetful functor from associative algebras to Lie algebras. If this map can be shown to be monic, then $F(M)$ is a subgroup of a free abelian group and is therefore free. All these functors commute with filtered colimits;

therefore, if we can show that the adjunction map is monic when M is free on a finite base, it is monic in general. Also, $F(M)$ is the free nonassociative algebra generated by M modulo the identities of a Lie algebra. The free nonassociative algebra is a graded algebra whose nth gradation is the sum of as many copies of $M^{(n)}$ as there are associations of n elements, which happens to be $\frac{1}{n+1}\binom{2n}{n}$, but is, in any case, finite. The identities are the two sided ideal generated by the homogeneous elements $x \otimes x$ and $x \otimes (y \otimes z) + z \otimes (x \otimes y) + y \otimes (z \otimes x)$. Thus $F(M)$ is a graded algebra; when M is finitely generated, so is the nth homogeneous component. Let $F_n(M)$ denote the sum of all the homogeneous components of $F(M)$ up to the nth. Let N be the kernel of $F(M) \to U(F(M)^{\mathrm{e}})$ and $N_n = N \cap F_n(M)$. Then N_n is finitely generated. If $N \neq 0$, then for some n, $N_n \neq 0$ since N is the union of them. Thus N_n is a non-zero finitely generated abelian group and it is a standard result that there is some prime p for which $\mathbf{Z}_p \otimes N_n \neq 0$. But $\mathbf{Z}_p$ is a field and both $()^{\mathrm{e}}$ and U commute with $\mathbf{Z}_p \otimes -$, so that reduces the question to the case of a field for which the Poincaré–Witt theorem, which gives the explicit form of the free basis, implies that the adjunction arrow is injective.

This finishes the case of a free module; projectives are readily handled as retracts of free modules. □

With this, Theorem 4.1 applies and shows that the cotriple resolution is homotopic to the one developed in [4].

6. Theorems on double complexes

This section contains the theorem on double complexes that is used to prove the main Theorem 4.1.

Let C' and C'' be differential (or differential graded) modules and suppose $f : C'' \to C'$ is a map between them. We begin by defining the *suspension* SC'' to be the same module (resp. differential graded module) with the negative of the boundary operator. In addition, in the graded case, the grading is to be raised by 1. That is, $(SC'')_n = C''_{n-1}$. Let $C = C' \oplus SC''$ with boundary operator

$$\begin{pmatrix} d' & f \\ 0 & -d'' \end{pmatrix}$$

where d' and d'' are the boundary operators in C' and C'', respectively. In the graded case, $C_n = C'_n \oplus C''_{n-1}$. It is easy to see that C is a differential (resp. differential graded) module and that we have an exact sequence

$$0 \to C' \to C \to SC'' \to 0.$$

It is almost easy to see that the connecting homomorphism $H(SC'') \to H(C')$ is essentially $H(f)$. In the graded case, $H_n(f) : H_n(C') \to H_n(C'') = H_{n-1}(SC'')$ which is the way the connecting homomorphism should go.

C is called the *mapping cone* of f and is frequently denoted C_f.

Proposition 6.1. *If C'' has trivial homology, then the inclusion $C' \subseteq C$ is a homology equivalence. If C'' is contractible, then $C' \subseteq C$ has a left inverse that is a homotopy inverse.*

Proof. The first is an immediate consequence of the exact triangle of homology. For the second, let $s'' : C'' \to C''$ be a contracting homotopy, which means that $s'' \circ d'' + d'' \circ s'' = 1$. The inclusion $i : C' \to C$ has matrix $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Let $j : C \to C'$ have matrix $(1 \quad f \circ s'')$. Then

$$d' \circ j = d'(1 \quad f \circ s'') = (d' \quad d' \circ f \circ s'') = (d' \quad f \circ d'' \circ s'')$$

and

$$j \circ d = (1 \quad f \circ s'')\begin{pmatrix} d' & f \\ 0 & -d'' \end{pmatrix} = (d' \quad f' - f \circ s'' \circ d'') = (d' \quad f \circ d'' \circ s'').$$

Thus j is a chain map. In the graded case it is also seen to preserve the grading. It is clear that $j \circ i = 1$. Let

$$s = \begin{pmatrix} 0 & 0 \\ 0 & s'' \end{pmatrix}.$$

We have

$$\begin{aligned} d \circ s + s \circ d &= \begin{pmatrix} d' & f \\ 0 & -d'' \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & s'' \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & s'' \end{pmatrix}\begin{pmatrix} d' & f \\ 0 & -d'' \end{pmatrix} \\ &= \begin{pmatrix} 0 & f \circ s'' \\ 0 & -d'' \circ s'' \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -s'' \circ d'' \end{pmatrix} \\ &= \begin{pmatrix} 0 & f \circ s'' \\ 0 & -1 \end{pmatrix} \end{aligned}$$

while

$$\begin{aligned} 1 - i \circ j &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1 \quad f \circ s'') \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & f \circ s'' \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & f \circ s'' \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

Thus $i \circ j$ is homotopic to the identity. □

It is often useful to recognize when an exact sequence of differential modules is a mapping cone sequence. Fortunately, the criterion is easy.

Proposition 6.2. *Let $0 \to C' \to C \to SC'' \to 0$ be a sequence of differential (resp. differential graded) modules. This is isomorphic to a mapping cone sequence if and only if it is split as a sequence of modules (resp. graded modules).*

Proof. We do this for the ungraded case. The graded case is similar. Since a mapping cone sequence is split, the necessity of the condition is clear. So suppose the sequence is split. Then, up to isomorphism, $C = C' \oplus C''$ and the inclusion and projection maps have matrices $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $(\,0 \;\; 1\,)$, respectively. The boundary operator has a matrix

$$\begin{pmatrix} e' & f \\ g & e'' \end{pmatrix}.$$

From

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} d' = \begin{pmatrix} e' & f \\ g & e'' \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

we conclude that $e' = d'$ and from

$$(\,0 \;\; 1\,) \begin{pmatrix} e' & f \\ g & e'' \end{pmatrix} = d''(\,0 \;\; 1\,)$$

we conclude that $g = 0$ and $e'' = d''$. Then from

$$\begin{pmatrix} d' & f \\ 0 & d'' \end{pmatrix}^2 = 0,$$

we see that $f \circ d'' + d' \circ f = 0$ so that $f : S^{-1}C'' \to C'$ is a chain map. □

In the following, use is actually made of properties of module categories. The property in question, that homology commutes with direct limits along chains is a consequence of the fact that in module categories filtered colimits commute with finite limits and, therefore, a filtered colimit of monomorphisms is a monomorphism. This is the property that Grothendieck later called AB5 is his famous "Tohôku" paper [6].

Corollary 6.3. *Let*

$$C_0 \to C_1 \to \cdots \to C_n \to \cdots$$

be a sequence of differential (resp. differential graded) modules. Suppose that C is the colimit of the sequence. If each $C_n \to C_{n+1}$ is a homology equivalence, then so is each $C_n \to C$.

Proof. This is an immediate consequence of the fact that homology commutes with filtered colimits. □

Theorem 6.4. *Let*

$$C_0 \to C_1 \to \cdots \to C_n \to \cdots$$

be a sequence of differential (resp. differential graded) modules. Suppose that C is the colimit of the sequence. If each $C_n \to C_{n+1}$ has a left inverse that is also a homotopy inverse, then the same is true for each $C_n \to C$.

Proof. Suppose that $f_n^m : C_m \to C_n$ denotes the composite arrow for $n \geq m$. Let $g_m^n : C_n \to C_m$, for $n \geq m$ be the composite of the left inverses, homotopy inverses that are known to exist. Then $g_m^n \circ f_n^m = 1$. For each n, there is a $h_n : C_n \to C_n$ such that $d \circ h_n + h_n \circ d = f_n^{n-1} \circ g_{n-1}^n$.

Lemma 6.5. *There is a sequence of maps $k_n : C_n \to C_n$ such that $d \circ k_n + k_n \circ d = 1 - f_n^0 \circ g_0^n$ and $k_n \circ f_n^{n-1} = f_n^{n-1} \circ k_{n-1}$.*

Proof. Let $k_0 = 0$. Assuming we have defined k_m for $m < n$, let

$$k_n = h_n - h_n \circ f_n^{n-1} \circ g_{n-1}^n + f_n^{n-1} \circ k_{n-1} \circ g_{n-1}^n.$$

Then we have

$$\begin{aligned}
d \circ k_n + k_n \circ d &= d \circ h_n - d \circ h_n \circ f_n^{n-1} \circ g_{n-1}^n + d \circ f_n^{n-1} \circ k_{n-1} \circ g_{n-1}^n \\
&\quad + h_n \circ d - h_n \circ f_n^{n-1} \circ g_{n-1}^n \circ d + f_n^{n-1} \circ k_{n-1} \circ g_{n-1}^n \circ d \\
&= d \circ h_n - d \circ h_n \circ f_n^{n-1} \circ g_{n-1}^n + f_n^{n-1} \circ d \circ k_{n-1} \circ g_{n-1}^n \\
&\quad + h_n \circ d - h_n \circ d \circ f_n^{n-1} \circ g_{n-1}^n + f_n^{n-1} \circ k_{n-1} \circ d \circ g_{n-1}^n \\
&= (d \circ h_n + h_n \circ d)(1 - f_n^{n-1} \circ g_{n-1}^n) \\
&\quad + f_n^{n-1} \circ (d \circ k_{n-1} + k_{n-1} \circ d) \circ g_{n-1}^n \\
&= (1 - f_n^{n-1} \circ g_{n-1}^n)^2 + f_n^{n-1} \circ (1 - f_{n-1}^0 \circ g_0^{n-1}) \circ g_{n-1}^n \\
&= 1 - f_n^{n-1} \circ g_{n-1}^n + f_n^{n-1} \circ g_{n-1}^n - f_n^{n-1} \circ f_{n-1}^0 \circ g_0^{n-1}) \circ g_{n-1}^n \\
&= 1 - f_n^0 \circ g_0^n
\end{aligned}$$

and

$$\begin{aligned}
k_n \circ f_n^{n-1} &= h_n \circ f_n^{n-1} - h_n \circ f_n^{n-1} \circ g_{n-1}^n \circ f_n^{n-1} + f_n^{n-1} \circ k_{n-1} \circ g_{n-1}^n \circ f_n^{n-1} \\
&= h_n \circ f_n^{n-1} - h_n \circ f_n^{n-1} + f_n^{n-1} \circ k_{n-1} \\
&= f_n^{n-1} \circ k_{n-1}
\end{aligned}$$

as required. □

By an obvious induction, we have that for $m < n$, $k_n \circ f_n^m = f_n^m \circ k_m$.

We now let $f^n : C_n \to C$ be the standard map to the colimit. Define $g_n : C \to C_n$ by

$$g_n \circ f^m = \begin{cases} f_m^n & \text{if } m \le n, \\ g_m^n & \text{if } m > n, \end{cases}$$

and $k : C \to C$ by $k \circ f^n = f^n \circ k_n$. We must show that these are compatible families. The first is a matter of considering cases and is left for an exercise. For the second, we have, for $m < n$, $f^n \circ k_n \circ f_n^m = f^n \circ f_n^m \circ k_m = f^m \circ k_m$ as required. By definition, $g_n \circ f^n = 1$. To calculate $d \circ k + k \circ d$ we compose with f^n:

$$\begin{aligned} d \circ k \circ f^n + k \circ d \circ f^n &= d \circ f^n \circ k_n + k \circ f^n \circ d = f^n \circ d \circ k_n + f^n \circ k \circ d \\ &= f^n \circ (d \circ k_n + k_n \circ d) = f^n \circ (1 - f_n^0 \circ g_0^n) \\ &= f^n - f^0 \circ g_0^n = f^n - f^0 \circ g_0^n \circ g_n \circ f^n \\ &= f^n - f^0 \circ g_0 \circ f^n = (1 - f^0 \circ g_0) \circ f^n. \qquad \square \end{aligned}$$

The hypotheses for this theorem may be too strong, but some hypothesis, beyond that of each $C_n \to C_{n+1}$ having a homotopy inverse, is needed. We give an example to show this. Note first that any morphism between contractible complexes is a homotopy equivalence; the 0 map in the opposite direction is a homotopy inverse. In addition, if a map from a contractible complex to another complex is invertible, then the second complex is also contractible. Thus it is sufficient to exhibit a sequence of contractible objects whose colimit is not contractible. We let $C_n \to C_{n+1}$ be the map from the left column to the right column of

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ \mathbf{Z}^n \arrow[r] \arrow[d] & \mathbf{Z}^{n+1} \arrow[d] \\ \mathbf{Z}^{n+1} \arrow[r, "\binom{0}{I}"] \arrow[d, "(1\ 2\ \cdots\ 2^n)"'] & \mathbf{Z}^{n+2} \arrow[d, "(1\ 2\ \cdots\ 2^{n+1})"] \\ \mathbf{Z} \arrow[r, "2"] \arrow[d] & \mathbf{Z} \arrow[d] \\ 0 & 0 \end{tikzcd}$$

The unlabeled vertical maps are simply the inclusion of the kernels and map between them is the induced map from one kernel to the other. The colimit of this sequence is the complex

$$0 \to \sum_{\aleph_0} \mathbf{Z} \to \sum_{\aleph_0} \mathbf{Z} \to 2^{-1}\mathbf{Z} \to 0.$$

(Here $2^{-1}\mathbf{Z}$ is the subring of the rationals generated by $\mathbf{Z}$ and $\frac{1}{2}$; equivalently, it is the subgroup of the additive group rationals of all $n/2^k$.) Although each of the C_n is contractible, the colimit sequence is not.

One can use this example to show that the limit (as opposed to the colimit) of a sequence of acyclic complexes is not acyclic. In fact the limit of the complex formed by homming the C_n into $\mathbf{Z}$ is not acyclic, while each of the constituent complexes is in fact contractible.

We apply Corollary 6.3 as follows. Consider a double complex $C_{\bullet\bullet}$:

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & C_{nm} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{0m} \arrow[r] \arrow[d] & C_{-1,m} \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & C_{n0} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{00} \arrow[r] \arrow[d] & C_{0,-1} \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & C_{n,-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{0,-1} \arrow[r] \arrow[d] & C_{-1,-1} \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 & 0 &
\end{tikzcd}$$

Let $T_m = T_m(C_{\bullet\bullet})$ be the double complex truncated above the mth row. Also let R_m be the mth row of the complex with the negative of the boundary operator and with the grading reduced by 1, so that the degree of the elements of C_{nm} have degree $n + m - 1$ as elements of R_m. If d denotes the horizontal and ∂ denotes the vertical boundary operators in the double complex, then the identity $d \circ \partial + \partial \circ d = 0$ implies that $d \circ \partial = \partial \circ (-d)$ so that $\partial : R_m \to T_{m-1}$ is a chain map. Its mapping cone is easily seen to be T_m and the mapping cone sequence is

$$0 \to T_{m-1} \to T_m \to R_m \to 0.$$

We then conclude,

Theorem 6.6. *Let $C_{\bullet\bullet}$ be a double complex as above. Suppose that every row R_m, $m \geq 0$ is acyclic (resp. contractible). Then the inclusion of $R_{-1} \to C_{\bullet\bullet}$ is a homology (resp. homotopy) equivalence.*

Corollary 6.7. *Let $C_{\bullet\bullet}$ be a double complex as above. Suppose that every row except the bottom and every column except the right are acyclic (resp. contractible). Then the bottom row and the rightmost column are homologous (resp. homotopic).*

Appendix. Beck modules

Although Beck's definition of module that we use here is widely known among category theorists, it does appear to have ever been published. In response to a suggestion of the referee, I give a brief exposition of the essential details.

The definition itself is simple. As mentioned in 2.1, if A is an object of the category $\mathscr{A}$, an A-module is an abelian group object of the slice category $\mathscr{A}/A$. What we want to indicate here is how, for familiar categories, associative algebras, commutative associative algebras, Lie algebras and groups, this definition reduces to two sided modules in the first case and left modules in the other three. We give many of the details in the first case and leave them to the reader for the other three, aside from a brief discussion of why you get just left modules.

A.1. Beck modules over an associative algebra. We fix a commutative ring K. All algebras will be assumed to be unital K-algebras, all modules will be assumed to be over K and all maps K-linear, without further mention. Let A be an algebra and suppose that $B \to A$ is an abelian group object in the category $\mathscr{A}/A$, where $\mathscr{A}$ is the category of associative algebras. What we are aiming to show is that, as a K-module, $B \cong A \times M$, and that the multiplication is given by

$$(a,m)(a',m') = (aa', am' + ma')$$

for a well determined two sided A-module M. We will also show that for any other $C \to A$ in $\mathscr{A}/A$, the abelian group of homomorphisms $\mathscr{A}/A(C \to A, B \to A)$ is just the group of derivations $\mathrm{Der}(C,M)$.

In order to have a group structure, there has to be first a zero map $z : A \to B$ which, to be a map over A, splits $B \to A$. Thus, as K-modules at least, $B \cong A \times M$, where $M = \ker(B \to A)$. We may suppose that $B = A \times M$ and the map to A are projections on the first coordinate. Since that projection is a ring homomorphism, it follows that the product is given by a formula of the form

$$(a,m)(a',m') = (aa', t(a,m,a',m')),$$

where t is some function. The distributive law implies that

$$t(a,m,a',m') = t(a,0,a',0) + t(a,0,0,m') + t(0,m,a',0) + t(0,m,0,m').$$

Since z is an algebra homomorphism, it follows from $z(aa') = z(a)z(a')$ that $(a,0)$ $(a',0) = (aa',0)$ so that $t(a,0,a',0) = 0$. If we write $t(a,0,0,m') = am'$, $t(0,m,a',0) = ma'$ and $t(0,m,0,m') = mm'$, the multiplication formula now reads $(a,m)(a',m') = (aa', am' + ma' + mm')$. It is an elementary calculation using associativity of multiplication to see that this right and left action of A on M makes the latter into a two sided A-module.

So far, we have used only the zero map. An abelian group object needs an "addition" in the form of an algebra homomorphism we denote $* : B \times_A B \to B$. Elements of $B \times_A B$ can be represented either as pairs of pairs $((a,m),(a',m'))$ or simply as 3-tuples (a,m,m'). We will denote $(a,m_1) * (a,m_2)$ by $(a,m_1,m_2)^*$. Since $*$ is a map over A, it follows that it must have the form $(a,m_1,m_2)^* = (a, s(a,m_1,m_2))$. The fact that $*$ is additive implies that s has the form $s(a,m_1,m_2) = s_0(a)+s_1(m_1)+s_2(m_2)$. Since z is the zero map, it follows that for any $a \in A$, $(a,m) * (a,0) = (a,0) * (a,m) = (a,m)$

from which we see that $s_0(a) = 0$ and $s_1(m) = s_2(m) = m$. Thus $(a, m_1, m_2)^* = (a, m_1 + m_2)$. The fact that $*$ preserves multiplication implies that $((1, m_1, 0)(1, 0, m_2))^* = (1, m_1, 0)^*(1, 0, m_2)^*$ which gives $(1, m_1 + m_2) = (1, m_1 + m_2 + m_1m_2)$ so that $m_1m_2 = 0$. This verifies the claim about the multiplication in B.

Now let $f : C \to A$ be an object of the slice category. A map $g : C \to B = A \times M$ over A must have first coordinate f and second coordinate a map we call $d : C \to M$. From the fact that g is additive, it follows that d is and the fact that d is multiplicative implies that

$$(g(cc'), d(cc')) = (gc \cdot gc', gc \cdot dc' + dc \cdot gc')$$

which means that $d(cc') = gc \cdot dc' + dc \cdot gc'$, which is the definition of a derivation of $C \to A$ into an A-module (with the induced action of C on A).

A.2. Beck modules in other categories. The arguments in other categories are quite similar. It may come as some surprise that you get just left modules (alternately, right modules) over groups, but that is what happens. If M is the kernel of the map $\Pi \to \pi$ that has the structure of an abelian group object in the category of groups over π, then π acts on M by $xm = z(x)mz(x)^{-1}$ where z is the zero section as before. This element (and not $z(x)m$ as in the case of associative algebras) is in the kernel of $\Pi \to \pi$ and that is where the structure comes from. In the case of commutative rings, you would appear to get two sided modules, but the fact that the larger ring is commutative forces the operations on the two sides to coincide. Similar things happen in the case of Lie algebras, where they differ by a sign.

References

[1] M. Barr and J. Beck, Acyclic models and triples, in: Proc. Conf. on Categorical Algebra (Springer, Berlin, 1966) 336–343.

[2] M. Barr and C.F. Wells, Triples, Toposes and Theories (Springer, Berlin, 1984).

[3] J. Beck, Triples, algebras and cohomology, Ph.D. Thesis, Columbia University, 1967.

[4] H. Cartan and S. Eilenberg, Homological Algebra (Princeton University Press, Princeton, 1956).

[5] P. Gabriel and F. Ulmer, Lokal präsentierbar Kategorien, Lecture Notes in Mathematics, Vol. 221 (Springer, Verlag, Berlin, 1971).

[6] A. Grothendieck, Sur quelques points d'algèbre homologique, Tohôku Math. J. 2 (1957) 119–221.